On drift and entropy growth for random walks on groups

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preliminary version

1 Introduction

We consider symmetric random walks on groups induced by the measure μ. In this paper we assume that the support of μ is finite and generates the group. We consider two functions

$$H(n) = -\sum_{g\in G} (\mu^{*n}) \ln(\mu^{*n}(g))$$

and

$$L(n) = \mathrm{E}_{\mu^{*n}} l(g).$$

Here l denotes the word metric, corresponding to the fixed set of generators and μ^{*n} is the n-th convolution of μ. The function $H(n)$ is called the *entropy* and $L(n)$ is called the *drift* of the random walk. $H(n)$ measures how far the measure μ^{*n} removed from being uniformely distributed. $L(n)$ shows how fast (in average) the random walk is moving away from the origin.

It is known that $H(n)$ is asymptotically linear iff $L(n)$ is asymptotically linear ([9]) and iff the Poisson boundary of the random walk is nontrivial ([7]). In particular, it is so for any non-amenable group. On the other hand, for many examples (e.g. for any Abelian group) $L(n)$ is asymptotically $\sqrt{n}$.

Until recently the existence problem for groups with intermediate growth rate of $L(n)$ was open. First examples were found by the author in [3], [5]. In these examples $L(n) \asymp n^{1-\frac{1}{2^k}}$ (for any positive integer k) and $L(n) \asymp n/\ln(n)$.

In this paper we find new possibilities for the rate of $L(n)$. In particular, we show that $L(n)$ can be asymptotically equal to

$$\frac{n}{\ln(\ln(\dots\ln(n)\dots))}.$$

We also estimate the growth of the entropy for random walks on these groups.

The structure of the paper is the following. In section 2 we state that certain functions are concave. We use this auxiliary lemma in the next section.

In section 3 we consider a two-dimensional simple random walk. We find asymptotics of some class of functionals depending on local times of this two-dimensional random walk.

In section 4 we construct examples of groups and we apply results of the previous section to find asymptotics of the drift in these groups.

In section 5 we give some general estimates for the entropy $H(n)$. We apply these estimates to the examples considered in section 4. These examples shows that there are infinitely many possibilities for asymptotics of the entropy. They also show that the growth of the entropy (as well as of the drift) can be very close to linear and yet sublinear.

2 An auxiliary lemma

Lemma 1 *For $0 < \alpha < 1$ let*

$$T_{k,\alpha} = \underbrace{\exp(\exp ... \exp((4k)^{1/\alpha})..).}_{k}$$

1. *Let*
$$\tilde{L}_{k,\alpha}(x) = \frac{x}{\underbrace{(\ln(\ln(... \ln(x))...))^{\alpha}}_{k}}.$$
Then $\tilde{L}_{k,\alpha}(x)$ is concave on the segment $[T_{k,\alpha}, \infty)$.

2. *There exists a continuous increasing function $L_{k,\alpha} : [0, \infty) \to [0, \infty)$ and $X > 0$ such that $L_{k,\alpha}(x)$ is concave, $L_{k,\alpha}(0) = 0$ and for $x > X$ $L_{k,\alpha}(x) = \tilde{L}_{k,\alpha}(x)$*

3. *For any $n \geq 1$ the function*
$$\frac{1}{\underbrace{\ln(\ln(... \ln(n/x))...)^{\alpha}}_{k}}.$$
is concave on the segment $(0, \frac{n}{T_{k,\alpha}}]$.

Proof. We prove this lemma in the Appendix.

3 Some functionals of two-dimensional random walk

We say that the random walk is simple if μ is equidistributed. In this section we consider a simple random walk on Z^2. Let $b_z^{(n)}$ be the number of times the

random walk has visited the element z ($z \in Z^2$) up to the moment n. Let $R^{(n)}$ be the number of different elements visited until the moment n. ($R^{(n)}$ is called *range* of the random walk). First we formulate a simple property of the range.

Lemma 2 *There exist $q_1, q_2 > 0$ such that for any $n > 1$*

$$\Pr[R^{(n)} \geq q_1 \frac{n}{\ln(n)}] \geq q_2.$$

Proof. Let $\tilde{R}^{(n)} = R^{(n)} \frac{\ln(n)}{n}$. Note that

$$\mathrm{E}[\tilde{R}^{(n)}] \asymp 1,$$

since (see for example [3])

$$\mathrm{E}[R^{(n)}] \asymp \frac{n}{\ln(n)}.$$

Hence it suffices to show that there exists C such that

$$\sigma^2[\tilde{R}^{(n)}] \leq C.$$

In the proof of the theorem 1 §4 [8] it is shown that for any random walk on a lattice

$$\sigma^2[R^{(n)})] \leq 2\mathrm{E}[R^{(n)}]\mathrm{E}[R^{(n-[n/2])}+R^{(n/2)}-R^{(n)}]+\mathrm{E}[R^{(n)}] \leq 6(\mathrm{E}[R^{(n)}])^2+\mathrm{E}[R^{(n)}].$$

Hence

$$\sigma^2[\tilde{R}^{(n)}] \leq 6 + \frac{1}{\mathrm{E}[R^{(n)}]}.$$

This completes the proof of the lemma.

Lemma 3 *Let $T > 0$ and f be a concave strictly increasing function such that*

1. *For any $C > 1$ $Cf(x) \geq f(Cx)$,*
2. *$f(0) = 0$ and there exists x such that $f(x) > 1$,*
3. *For any $n \geq 1$ $xf(n/x)$ is concave on the segment $(0, n/T]$.*

Consider a simple random walk on $\mathbb{Z}^2$. Then there exists $K > 0$ such that

$$\mathrm{E}\left(\sum_{z \in \mathbb{Z}^2} f(b_z^{(n)})\right) \leq Kf(\ln(n))\frac{n}{\ln(n)}.$$

Proof. Note that

$$\sum f\left(\sum b_z^{(n)}\right) \leq f(n/R^{(n)})R^{(n)},$$

since f is concave. Hence

$$\mathrm{E}\left[\sum f\left(\sum b_z^{(n)}\right)\right] \leq \mathrm{E}[f(n/R^{(n)})R^{(n)}] =$$

$$\mathrm{E}[f(n/R^{(n)})R^{(n)}|0, R^{(n)} < n/T] + \mathrm{E}[f(n/R^{(n)})R^{(n)}|R^{(n)} \geq n/T].$$

Let M be the maximum of f on the segment $[0,T]$, that is $M = f(T)$, since f is an increasing function. Then the second term is not greater than

$$\mathrm{E}[MR^{(n)}|R^{(n)} \geq n/T] \leq M\mathrm{E}[R^{(n)}] \asymp \frac{n}{\ln(n)}.$$

But for n large enough we have $f(\ln(n)) > 1$ and hence

$$\frac{n}{\ln(n)} \leq f(\ln(n))\frac{n}{\ln(n)}.$$

Now we want to estimate the first term. Since $xf(n/x)$ is concave on $(0, n/T]$ we have

$$\mathrm{E}[f(n/R^{(n)})R^{(n)}|0, R^{(n)} < n/T] \leq$$

$$\mathrm{E}[R^{(n)}|0 < R^{(n)} < n/T]f\left(\frac{n}{\mathrm{E}[R^{(n)}|0, R^{(n)} < n/T]}\right) \leq$$

$$E[R^{(n)}]f(\frac{n}{\mathrm{E}[R^{(n)}]}) \asymp f(\ln(n))\frac{n}{\ln(n)}.$$

Then there exists $K > 0$ such that

$$\mathrm{E}\left(\sum_{z\in\mathbb{Z}^2} f(b_z^{(n)})\right) \leq Kf(\ln(n))\frac{n}{\ln(n)}.$$

This completes the proof of the lemma.

The following lemma gives an estimate from the other side.

Lemma 4 *Let f be a strictly increasing function on $[0,\infty)$ such that $f(0) = 0$ and for any $C > 1$ $f(Cx) \leq Cf(x)$. Then for n large enough and for some positive ϵ we have*

$$\mathrm{E}\left[\sum_{z\in\mathbb{Z}^2} f(b_z^{(n)})\right] \geq \epsilon f(\ln(n))\frac{n}{\ln(n)}.$$

Proof. Note that for any $\varepsilon_1 > 0$ there exists $K > 0$ such that for n large enough

$$\Pr\left[b_0^{(n)} \geq K\ln(n)\right] \geq 1 - \varepsilon_1$$

(This follows from theorem 1 [6]). Let $n \geq 4$ and let $m = [n/2]$. Since $m > 1$ lemma 2 implies that

$$\Pr[R^{(m)} \geq q_1 \frac{m}{\ln(m)}] \geq q_2.$$

Let $x_1^{(n)}, ..., x_s^{(n)}$ be different points visited by the random walk up to the moment n, enumerated in the order of visiting.

Let $\beta_i^n = b_{x_i^{(n)}}^{(n)}$.

Take ε_1 such that $\varepsilon_1 \leq 1/2q_2$

Note that for any $0 \leq i \leq q_1 \frac{m}{\ln(m)}$

$$\Pr\left[\beta_i^{(n)} \geq K\ln(m)\right] \geq \Pr[R^{(m)} \geq i] \Pr\left[\beta_i^{(n)} \geq K\ln(m)\right] \geq$$

$$q_2 \Pr\left[\beta_i^{(n)} \geq K\ln(m) | R^{(m)} \geq i\right] \geq$$

$$q_2 \Pr\left[\beta_0^{(m)} \geq K\ln(m)\right] \Pr\left[R^{(m)} \geq i\right] \geq$$

$$q_2^2(1 - \varepsilon_1) \geq q_2^2/2$$

Hence for any n large enough

$$\mathrm{E}\left[\sum_{z \in \mathbb{Z}^2} f(b_z^{(n)})\right] \geq \frac{q_2^2}{2} f(K\ln([n/2])) q_1 \frac{n/2}{\ln(n/2)} \asymp f(\ln(n)) \frac{n}{\ln(n)}.$$

Corollary 1 *1. Let $L_{k,\alpha}(x)$ be the function defined in Lemma 1 ($0 < \alpha < 1$). Then*

$$\mathrm{E}\left[\sum_{z \in \mathbb{Z}^2} L_{k,\alpha}(b_z^{(n)})\right] \asymp L_{k+1,\alpha}(n).$$

2. Let $f(x) = x^\alpha (0 < 1 < \alpha)$ then

$$\mathrm{E}\left[\sum_{z \in \mathbb{Z}^2} f(b_z^{(n)})\right] \asymp n/\ln(n)^{(1-\alpha)}$$

Proof. This corollary follows from lemma 2 and lemma 3 since for $n > N$

$$\frac{n}{\ln(n)} L_{k,\alpha}(\ln(n)) \asymp \frac{n}{\ln(n)} \frac{\ln(n)}{(\ln(\ln ... \ln(n)...))^{\alpha}} \asymp L_{k+1,\alpha}(n)$$

and

$$\frac{n}{\ln(n)} \ln(n)^{1-\alpha} = n/\ln(n)^{\alpha}.$$

4 Main result

First we recall the definition of the wreath product.

Defintion 1 *The wreath product of C and D is a semidirect product C and $\sum_C D$, where C acts on $\sum_C D$ by shifts: if $c \in C$, $f : C \to D, f \in \sum_C D$, then $f^c(x) = f(xc^{-1}), x \in C$. Let $C \wr D$ denote the wreath product.*

Lemma 5 *Let $a_1, a_2, ..., a_k$ generate A. Consider the symmetric simple random walk on A corresponding to this set of generators. Let*

$$L(n) = L_n^A.$$

Then for some simple random walk on $B = \mathbb{Z}^2 \wr \mathrm{A}$

$$L_n^B \asymp \mathrm{E} \sum_{z \in \mathbb{Z}^2} L(b_z^{(n)})$$

Proof. The proof of this lemma is similar to that of lemma 3 in [5]. For any $a \in A$ $\tilde{a}^e$ denotes the function from $\mathbb{Z}^2$ to A such that $\tilde{a}^e(0) = a$ and $\tilde{a}^e(x) = e$ for any $x \neq 0$. Let $a^e = (e, \tilde{a}^e)$. Let e_1', e_2' be the standard generators of $\mathbb{Z}^2$ and $e_1 = (e_1', e)$, $e_2 = (e_2', e)$.

Consider the following set of generators of B:

$$(a_j^e)^p e_s (a_n^e)^q,$$

$p, q = 0, 1$ or -1, $s = 1$ or 2 and $1 \leq j, n \leq k$.

Consider the simple random walk on B, corresponding to this set of generators.

Suppose that this random walk hits (a, f) at the moment n. Let $l_A(f(z)) = c_z$. Then

$$\frac{1}{2} \sum c_z \leq l(a, f) \leq 2(\sum c_z + R)$$

To see this note that multiplying by one of the generators changes the value of f in at most 2 points and that two multiplyings suffice if we want to change the value of f in a standing at the point a.

Hence

$$\frac{1}{2}\mathrm{E}\left[\sum c_z\right] = \frac{1}{2}\sum \mathrm{E}[c_z] \leq \mathrm{E}[l(a,f)] \leq 2(\sum \mathrm{E}[c_z] + \mathrm{E}[R])$$

The projection of this random walk on $\mathbb{Z}$ is simple and symmetric. Note that in any $i \in \mathbb{Z}^2$ there is a simple random walk on A. The measure that defines this random walk is supported on $a_i (1 \leq i \leq n)$, e, and inverses of these elements. It is uniformely distributed on a_i and their inverses. Let $\hat{L}_n^A$ be the drift that corresponds to this new measure. It is clear that

$$\hat{L}_n^A \asymp L_n^A.$$

Note that

$$\begin{aligned}
\mathrm{E}[c_z | b_z = b, z \neq 0, a \neq z] &= \hat{L}_{2b}^A \\
\mathrm{E}[c_z | b_z = b, z = 0, a \neq z] &= \hat{L}_{2b-1}^A \\
\mathrm{E}[c_z | b_z = b, z \neq 0, a = z] &= \hat{L}_{2b-1}^A \\
\mathrm{E}[c_z | b_z = b, z = 0, a = z] &= \hat{L}_{2b-2}^A
\end{aligned}$$

Hence

$$\mathrm{E}\left[\min(\hat{L}_{2b}^A, \hat{L}_{2b-1}^A, \hat{L}_{2b-2})\right] \leq \mathrm{E}[c_z | b_z = b] \leq \mathrm{E}\left[\max(\hat{L}_{2b}^A, \hat{L}_{2b-1}^A, \hat{L}_{2b-2})\right]$$

There exist $C_2, C_3 > 0$ such that

$$C_2 L(n) \leq \hat{L}_{2n-2}^A, \hat{L}_{2n-1}^A, \hat{L}_{2n}^A \leq C_3 L(n)$$

Hence

$$C_2 \mathrm{E}[\sum_{i \in Z^2} L(b_z)] \leq \mathrm{E}[c_z] \leq C_3 \mathrm{E}[\sum_{i \in Z^2} L(b_z)]$$

This completes the proof of the lemma.

Theorem 1 *1. Let F be a finite group. Consider the following groups that are defined recurrently*

$$G_1 = \mathbb{Z}^2 \wr \mathrm{F}; \mathrm{G_{i+1}} = \mathbb{Z}^2 \wr \mathrm{G_i}.$$

Then for some simple random walk on G_i and for any n large enough we have

$$L_n^{G_i} \asymp \frac{n}{\underbrace{\ln(\ln \ldots \ln n) \ldots)}_{k}}$$

2. *Consider the following groups that are defined recurrently*

$$F_1 = \mathbb{Z}; \mathrm{F_{i+1} = \mathbb{Z} \wr F_i}$$

and let

$$H_{1,i} = \mathbb{Z}^2 \wr \mathrm{F_i; H_{j+1,i} = \mathbb{Z}^2 \wr H_{j,i}}$$

Then for some simple random walk on $H_{j,i}$ and for any n large enough we have

$$L_{H_{j,i}}(n) \asymp \frac{n}{\sqrt[2^i]{\underbrace{\ln(\ln(\ldots \ln}_{j}(n)\ldots))}}.$$

Proof.

1. We prove the theorem by induction on i. Base $i = 1$. In this case $G_i = \mathbb{Z}^2 \wr \mathrm{F}$ and $L(n)$ is asymptorically equal to $\frac{n}{\ln(n)}$ ([3]) Induction step follows from the previous lemma and corollary 1.

2. We prove the statement by induction on j. For $H_{1,i} = F_i$ the asymptotics of the drift is found in [5]. Is is proven there that for some random walk on F_i

   $$L_{F_i}(n) \asymp n^{1-\frac{1}{2^k}}.$$

   The induction step follows from the previous lemma and colollary 1.

5 Estimates of the entropy

In this section we give estimates for the entropy of a random walk. It is known (see [2]) that for a wide class of measures on nilpotent groups $H(n) \asymp \ln(n)$. As it was mentioned before $H(n)$ is asymptotically linear for any nonamenable group. In this section we study intermediate examples.

Let $v(n)$ be the growth function of the group (that is $v(n) = \#\{g \in G : l(g) \leq n\}$). Let

$$l = \lim_{n\to\infty} L(n)/n,$$

$$h = \lim_{n\to\infty} H(n)/n,$$

$$v = \lim_{n\to\infty} \ln(v(n))/n.$$

(See for example [1] for the proof that these limits exist.) It is known (see [1]) that

$$h \leq vl.$$

The following lemma generalizes this fact.

Lemma 6 *For any $\varepsilon > 0$ there exists $C > 0$ such that*

$$H(n) \leq (v+\varepsilon)L(n) + \ln(n) + C.$$

Proof. Let $a_i^{(n)} = \Pr_{\mu^{*n}} l(g)$. Then by definition

$$L(n) = \sum_{i=0}^{n} i a_i^{(n)}.$$

Note that

$$H(n) \leq \sum_{i=1}^{n} a_i^{(n)} \ln(v(i)/a_i^{(n)}) =$$

$$\sum_{i=1}^{n} a_i^{(n)} \ln(v(i)) + \sum_{i=1}^{n} a_i^{(n)}(-ln(a_i^{(n)})) \leq \sum_{i=1}^{n} a_i^{(n)} \ln(v(i)) + \ln(n).$$

For any $\varepsilon > 0$ there exists $K > 0$ such that $v(i) \leq K(v+\varepsilon)^i$. Hence

$$H(n) \leq \sum_{i=1}^{n} a_i^{(n)}(i(v+\varepsilon) + \ln(K)) + \ln(n) =$$

$$(v+\varepsilon)\sum_{i=1}^{n} a_i^{(n)} i + \ln(K)) + \ln(n) = (v+\varepsilon)L(n) + \ln(K)) + \ln(n).$$

Another lemma estimates the entropy from the other side:

Lemma 7 *1. There exists $C > 0$ such that*

$$H(n) \geq C\mathrm{E}_{\mu^{*n}} l^2(g)/n - \ln(n) \geq CL^2(n)/n - \ln(n)$$

2. There exists $K > 0$ such that

$$L(n) \leq K\sqrt{n(\ln(v(n)) + \ln(n))}.$$

Proof.

1. Let $p_n(x)$ be the probability to hit x after n steps. In [9] it is shown that there exist $K_1, K_2 > 0$ such that for any x and n

$$p_n(x) \leq K_1 n^{3/4} \exp(-K_2 l(x)^2/n) \leq K_1 n \exp(-K_2 l(x)^2/n).$$

Note that then

$$-\sum_{g\in G:l(g)=i} \mu^{*n}(g)\ln(\mu^{*n}(g)) \geq (-\ln(K_1) - \ln(n) + K_2 i^2/n) a_i^{(n)}$$

Hence

$$H(n) \geq -\ln(K_1) - \ln(n) + K_2 \sum_{i=0}^{n} i^2/na_i^{(n)} =$$

$$-\ln(K_1) - \ln(n) + K_2/n\mathrm{E}_{\mu^{*n}} l^2(g) \geq C/n\mathrm{E}_{\mu^{*n}} l^2(g) - \ln(n).$$

The last inequality follow from the fact that for some $C_2 > 0$ $H(n) \geq C_2$.

2. This follows from the first part of the lemma, since $H(n) \leq \ln(v(n))$.

As a corollary from the two previous lemmas we get the following theorem.

Theorem 2 *Let G_i be the groups defined in theorem 1. Then for some random walk on G_i we have*

$$K_1 n/\underbrace{\ln(\ln\ldots\ln}_{i}(n)\ldots)^2 \leq H_{G_i}(n) \leq K_2 n/\underbrace{\ln(\ln\ldots\ln}_{i}(n)\ldots)$$

for some positive constansts K_1 and K_2. In particular, all G_i have different asymptotics of the entropy.

A Proof of the auxiliary lemma

In the appendix we give the proof of lemma 1.

1. Let $m_{k,\alpha}(x) = (\underbrace{\ln(\ln\ldots\ln(x)\ldots)}_{k})^{\alpha}$. We want to prove that $x/m_{k,\alpha}(x)$ is concave on $[T_{k,\alpha}, \infty)$. Note that

$$(x/m_{k,\alpha})'' = \frac{-xm''_{k,\alpha}m^2_{k,\alpha} - 2m'_{k,\alpha}m^2_{k,\alpha} + 2xm_{k,\alpha}(m'_{k,\alpha})^2}{m^4_{k,\alpha}}.$$

Since $m_{k,\alpha}(x) > 0$ on $(0, n/T_{k,\alpha}]$ it suffices to prove that

$$2x(m'_{k,\alpha})^2 - xm_{k,\alpha}m''_{k,\alpha} \leq 2m'_{k,\alpha}m_{k,\alpha}.$$

To prove this we will show that

$$2x(m'_{k,\alpha})^2 < 1/2m_{k,\alpha}m'_{k,\alpha}$$

and that

$$m_{k,\alpha}(-m''_{k,\alpha})x \leq 1.5m'_{k,\alpha}m_{k,\alpha}.$$

Note that

$$m'_k(x) = \frac{\alpha}{x\underbrace{\ln(x)\ln(\ln(x))\ldots\ln(\ln(\ldots\ln(x)\ldots))}_{k-1} m^{1-\alpha}_{k,\alpha}}.$$

We see that $0 < m'_{k,\alpha}(x) < 1/x$ and we know that $m_{k,\alpha}(x) > 4$ ($x \in [T_{k,\alpha}, \infty)$). This proves the first inequality.

Now let

$$r_{k,\alpha}(x) = \alpha/m'_k = x \underbrace{\ln(x)\ln(\ln(x))...\ln(\ln(...\ln(x)...))}_{k-1} m_{k,\alpha}^{1-\alpha}.$$

Note that

$$r'_{k,\alpha}(x) = m_{k,\alpha}^{1-\alpha} m_{1,1}(x) m_{2,1}(x)...m_{k-1,1}(x)+$$
$$\sum_{i=1}^{k-1} r_{k,\alpha}(x)\frac{m'_{i,1}(x)}{m_{i,1}(x)} + r_{k,\alpha}\frac{(1-\alpha)m'_{k,\alpha}}{m_{k,\alpha}}.$$

Note that for $1 \le i \le k-1$ $m'_{i,1}(x) \le 1/x$ and that $m'_{k,alpha}(x) \le 1/x$. Since $m_{k,\alpha}(x) \ge 2k$ and for $1 \le i \le k-1$ $m_i(x) \ge 2k$ we get

$$r'_{k,\alpha} x \le 1.5 r_{k,\alpha}.$$

This implies that

$$(-m''_{k,\alpha})x = \alpha x r'_{k,\alpha}/r^2_{k,\alpha} \le \alpha 1.5/r_{k,\alpha} = 1.5 m'_{k,\alpha}.$$

So we have proven also the second inequality.

2. Let $\beta_k = \tilde{L}_{k,\alpha}(T_{k,\alpha})/(2T_{k,\alpha})$. Consider the function $y_k(x) = \beta_k x$. Note that $y_(T_{k,\alpha}) < \tilde{L}(T_{k,\alpha})$ and that for $x > X$ $y_(x) > \tilde{L}(x)$. Take maximal z such that $y_(z) = \tilde{L}(z)$. Let $L_{k,\alpha}(x) = y(x)$ if $0 \le x \le z$ and $L_{k,\alpha}(x) = \tilde{L}_{k,\alpha}(x)$ if $x \ge z$. Note that $y(x)$ is concave ($x \in [0,x]$), $\tilde{L}_k(x)$ is concave if $x > z$ and that $\tilde{L}'_{k,\alpha}(z) \le y'(z)$. This implies that $L_{k,\alpha}$ is concave.

3. We can assume without loss of generality that $\alpha = 1$ since x^α is a concave function. Let $g(x) = \underbrace{ln(ln...(ln(n/x)))}_{k}$. We want to proof that $1/g$ is concave on the given segment. Note that

$$(1/g)'' = \frac{-g''g^2 + 2g(g')^2}{g^4}.$$

Since g is positive on the given segment, we need to prove that

$$2(g')^2 v \le g'' g.$$

Let

$$h(x) = x \underbrace{\ln(n/x)\ln(\ln(n/x))...\ln(\ln...\ln(n/x)...))}_{k-1}.$$

Note that $g'(x) = -1/h$ and $g''(x) = h'/(h^2)$. We want to prove that $2 < gh'$. Since $g(x) > 2$ on the given segment, it suffices to note, that $h'(x) > 1$. This follows from the fact that $h(x)/x$ increases and greater than 1.

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